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Implementation of Listing's Law for a Tendon Driven Robot Eye

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***Abstract*— This paper presents a model for a tendon driven robot eye designed to emulate the actual saccadic and smooth pursuit movements performed by human eyes. Physiological saccadic motions obey the so called *Listing's Law* which constrains the admissible eye's angular velocities. The paper discusses conditions making possible to implement the *Listing's Law* on a purely mechanical basis, i.e. without active control.**

I. Introduction

Eye movements have been studied since the mid of the 19^{th} century. However, only during the past 20 years quantitative mathematical models have been proposed, and validated by experiments and clinical tests.

Saccades are a very important class of eye motions, [1]. During saccades the eye orientation is determined by a basic principle known as *Listing's Law*, [2], which establishes the amount of eye torsion for each direction of fixation. *Listing's Law* has been experimentally verified on humans and primates [2]– [5], and also found to be valid during other types of eye movements such as *smooth pursuit*, [6]. The geometric properties of *Listing's Law*, [2], [3], [7]– [9], have significant implications on the eye control mechanisms. In fact, recent anatomical advances, [10]– [14], suggest that the mechanics of the eye plant could play a significant role to implement *Listing's Law*, [9], [15]– [17].

The major goal of this paper is to present a model of the eye plant mechanics which ensures the possibility of implementing *Listing's Law* on a purely mechanical basis.

This result represents an important step to better understand the mechanical and control mechanisms implemented at biological level, but also a fundamental step to design humanoid robot eye devices. As a matter of fact, many eye-head robots have been designed in the past few years, e.g. [18]– [20], but little attention has been paid to emulate the actual mechanics of the eye. On the other hand, methodological studies in the area of modelling and control of human-like eye movements have been presented, [21]– [24].

In the following a realistic model of the eye-plant is presented and it is shown the possibility of implementing *Listing's Law* on a mechanical basis. The eye is modelled as a sphere actuated by tendons emulating the action of *extra-ocular muscles* routed through pulleys. Proper positioning of the pulleys and of the insertion points of the muscles on the eye-ball, allow to transform the action of the muscle forces into a torque constrained to generate Listing compatible motions. The analysis shows also that starting from a reference position, also known *primary position*, it is possible to reach any admissible (i.e. in the reach space), eye orientation with a Listing compatible trajectory. The analysis presented in this paper has been the basis for the development of an embedded robot eye presented in [25], [26].

The structure of the paper is the following. In section II *Listing's Law* is presented and some of its relevant geometric and kinematic properties discussed. A model of the eye plant is presented in section III. Then, in section IV the implications of the eye model on the space of actuating moments is discussed. In section V, the domain of Listing compatible eye orientations is determined and the existence of Listing compatible trajectories is eventually shown. Simulations experiments extending the analytical results are presented in section VI, as well as a short description of a prototype of robot eye built in agreement with the theory discussed in this paper.

II. Saccadic Movements and Listing's Law

Eye movements have the goal of optimizing visual perception, [27]. The way the eyes change their orientation may affect our perception of the world. In turn, it is widely accepted that visual feedback, as well as other sensory feedback (e.g. from vestibular system), play a major role in stimulating eye movements. Therefore, it is not surprising that to different vision strategies correspond significantly different types of eye motions. During saccades, for instance, the goal is that of reaching as fast as possible a target direction of fixation, while during *vestibulo-ocular reflex* (VOR) the main goal is to keep stable the image on the retina despite possible *external disturbances*.

In the following we will focus on saccadic motions, and introduce *Listing's Law* which specifies the eye's orientation during saccades.

Listing's Law. There exists a specific eye orientation with respect to the head, called *primary position*. During saccades any physiological eye orientation, with respect to the *primary*

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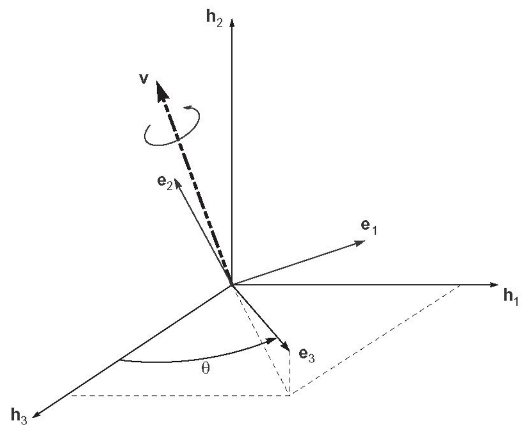

Fig. 1. Geometry of Listing compatible rotations: the finite rotation axis $\mathbf{v}$ is always orthogonal to $\mathbf{h_3}$.

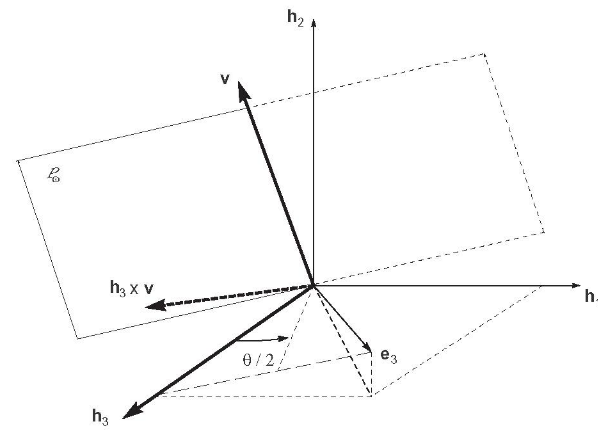

Fig. 2. Half angle rule geometry. The eye's angular velocity must belong to the plane $\mathcal{P}_\omega$ passing through axis $\mathbf{v}$.

position, can be described by a unit quaternion q whose (unit) rotation axis, $\mathbf{v}$, always belongs to a head fixed plane, $\mathcal{L}$. The normal to plane $\mathcal{L}$ is the eye's direction of fixation at the *primary position*.

Let $\langle h\rangle = \{\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}\}$, and $\langle e\rangle = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be respectively a head fixed and a eye fixed reference frames. Without loss of generality we can assume that $\mathbf{e_3}$ is the fixation axis of the eye, and that $\langle h\rangle \equiv \langle e\rangle$ at the *primary position*; then, $\mathcal{L} = span\{\mathbf{h_1}, \mathbf{h_2}\}$. Fig. 1 shows the geometry of Listing compatible rotations.

During saccades, at any time t, the *finite rotation* of the eye can be conveniently described by a unit quaternion:

$$q(t) = \left(\cos\frac{\theta(t)}{2}, \mathbf{v}(t)\sin\frac{\theta(t)}{2}\right), \tag{1}$$

where $\mathbf{v}(t) \in \mathcal{L}, |\mathbf{v}(t)| = 1$, and $\theta(t)$ is the rotation angle with respect to the *primary position*. The derivative of (1) is (omitting the time dependencies):

$$\dot{q} = \frac{1}{2}\,\tilde{\omega}\,q, \tag{2}$$

where quaternion $\tilde{\omega} = (0, \omega)$ and ω is the angular velocity of the eye. By expanding (2) we obtain:

$$\dot{q} = \frac{1}{2}\left(-(\omega\cdot\mathbf{v})\sin\frac{\theta}{2}, \omega\cos\frac{\theta}{2} + (\omega\times\mathbf{v})\sin\frac{\theta}{2}\right). \tag{3}$$

In order to guarantee that $\mathbf{v} \in \mathcal{L}$, the condition $\dot{\mathbf{v}} \in \mathcal{L}$, must be satisfied. Then, accordingly to (3) the following equality must hold:

$$\mathbf{h_3}\cdot\left[\omega\cos\frac{\theta}{2} + (\omega\times\mathbf{v})\sin\frac{\theta}{2}\right] = 0. \tag{4}$$

Expression (4) leads to the formula:

$$(\omega\cdot\mathbf{h_3}) = \omega\cdot(\mathbf{h_3}\times\mathbf{v})\tan\frac{\theta}{2}, \tag{5}$$

which states that two components of angular velocity vector of the eye must be constrained each other in order to ensure implementation of *Listing's Law*, while the third one (directed along the axis $\mathbf{v}$) can assume any value. In particular, ω must belong to a plane $\mathcal{P}_\omega$ passing through $\mathbf{v}$, and whose normal forms an angle of $\frac{\theta}{2}$ with axis $\mathbf{h_3}$, see Fig. 2. This property is directly implied by *Listing's Law*, and is usually called *Half Angle Rule*, [8].

The *Half Angle Rule* has important implications. First of all, although *Listing's Law* implies zero torsion of the eye during saccades, the eye's angular velocities in turn may have a torsional component (i.e. a component directed along axis $\mathbf{h_3}$). The second, and most important remark, is that ω is constrained to lay on a moving plane, $\mathcal{P}_\omega$ which is not fixed to the head, neither to the eye for its dependency from $\mathbf{v}$ and $\frac{\theta}{2}$. This fact poses important questions related to the control mechanisms required to implement the *Listing's Law*, also since there is no evidence of sensors in the eye plant capable to detect how $\mathcal{P}_\omega$ is oriented.

In the following a model explaining the feasibility of the implementation of the listing law on a purely mechanical basis is discussed. This basic result provides a formal proof to the claim that the mechanics of the eye plant could have a significant role in the implementation of *half angle rule* and *Listing's Law*, [15] – [17]. Furthermore, from a robotic perspective the proposed model provided important guidelines for the design of a human-like tendon driven robotic eye.

III. Eye Model

The eye-ball is assumed to be modelled as a homogeneous sphere of radius R, having 3 rotational *degrees of freedom* (DOFs) about its center, and actuated by the action of six *extra-ocular muscles* (EOMs), [30]. Accordingly with the rationale proposed in [9], and [15] only the four *rectii* EOMs are taken into account here, assuming negligible the role of the upper and lower *obliqui* muscles during saccades. Finally, the EOMs are modelled as non-elastic thin wires, [15], connected to pulling *force generators*, [22].

Starting from the *insertion points* (IPs) on the eye-ball, the EOMs are routed through head fixed *point-wise pulleys* (PPs), emulating the *soft-pulley tissue* discussed in [10] – [14]. The PPs are located on the rear of the eye-ball. In the following sections it will be shown that appropriate placement of the PPs and of the IPs has a fundamental role to implement the *Listing's Law* on a purely mechanical basis.

Let O be the eye's center, then the position of the PPs can be described by the vectors $\mathbf{p_i}$. At the *primary position* IPs can be described by the vectors $\mathbf{c_i}$ such that $|\mathbf{c_i}| = R$.

Let $rot(\mathbf{v}, \theta)$ be the operator rotating a generic vector about a unit vector $\mathbf{v}$ by an angle θ. Then, by Rodrigues Formula for a generic vector $\mathbf{x}$, we have:

$$rot(\mathbf{v},\theta)\mathbf{x} = (\mathbf{v}\cdot\mathbf{x})\mathbf{v} + (\mathbf{v}\times\mathbf{x})\sin\theta - \mathbf{v}\times(\mathbf{v}\times\mathbf{x})\cos\theta \quad (6)$$

When the eye is rotated about an axis $\mathbf{v}$ by an angle θ the position of the IPs can be expressed as:

$$\mathbf{r_i} = rot(\mathbf{v},\theta)\ \mathbf{c_i} \qquad \forall\ i = 1\ldots 4. \quad (7)$$

Each EOM is assumed to follow the shortest path from each IP to the corresponding pulley, [12]; the path of the each EOM, for any eye orientation, belongs to a plane defined by vectors $\mathbf{r_i}$ and $\mathbf{p_i}$. Therefore, the torque applied to the eye by the action of each EOM is given by:

$$\boldsymbol{\tau_i} = \tau_i \frac{\mathbf{m_i}}{|\mathbf{m_i}|} \qquad \forall\ i = 1\ldots 4, \quad (8)$$

where $\tau_i \geq 0$ is the magnitude of the pulling force generated by the $i-th$ EOM, while $\mathbf{m_i}$ is the normal to the EOM's plane defined as:

$$\mathbf{m_i} = \mathbf{r_i}\times\mathbf{p_i} \qquad \forall\ i = 1\ldots 4. \quad (9)$$

From expressions (8) and (9), it is clear that $|\mathbf{p_i}|$ does not affect the direction of $\mathbf{m_i}$ and the applied moment. Therefore, in the following, without loss of generality it is assumed that:

$$|\mathbf{p_i}| = |\mathbf{c_i}| \qquad \forall\ i = 1\ldots 4. \quad (10)$$

We assume also that $\mathbf{p_i}$ and $\mathbf{c_i}$ are symmetric with respect to the plane $\mathcal{L}$ which implies:

$$(\mathbf{v}\cdot\mathbf{c_i}) = (\mathbf{v}\cdot\mathbf{p_i}) \qquad \forall\ i = 1\ldots 4, \forall\ \mathbf{v}\in\mathcal{L} \quad (11)$$

By (11) and (7) the following equalities also hold:

$$(\mathbf{v}\cdot\mathbf{r_i}) = (\mathbf{v}\cdot\mathbf{p_i}) \qquad \forall\ i = 1\ldots 4, \forall\ \mathbf{v}\in\mathcal{L} \quad (12)$$

Finally we assume that:

$$(\mathbf{h_3}\cdot\mathbf{c_i}) = (\mathbf{h_3}\cdot\mathbf{c_j}) \qquad \forall\ i,j = 1\ldots 4, \quad (13)$$

and

$$(\mathbf{c_3}-\mathbf{c_1})\cdot(\mathbf{c_4}-\mathbf{c_2}) = 0. \quad (14)$$

The last two conditions state that IPs are symmetric with respect to the fixation axis.

Fig. 3 shows the relative position of IPs and PPs when the eye is in its *primary position*, assuming without loss of generality: $(\mathbf{c_3}-\mathbf{c_1})\times\mathbf{h_1} = 0$, and $(\mathbf{c_4}-\mathbf{c_2})\times\mathbf{h_2} = 0$.

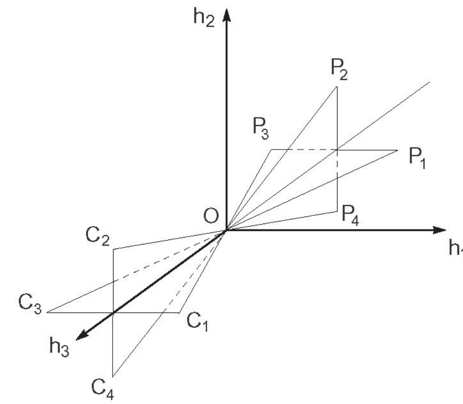

Fig. 3. Relative position of pulleys and insertion points when the eye is in the *primary position*.

IV. Geometric Properties of Vectors $\mathbf{m_i}$

In this section we prove that, under the assumptions made in the previous section, the vectors $\mathbf{m_i}$ belong to plane $\mathcal{P}_\omega$, defined in section II.

Lemma 1: Given two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ then:

$$rot(\mathbf{v},\theta)(\mathbf{a}\times\mathbf{b}) = rot(\mathbf{v},\theta)\mathbf{a}\times rot(\mathbf{v},\theta)\mathbf{b}$$

Proof: Obvious ∎

Given the vectors $\mathbf{p_i}$, $\mathbf{c_i}$ and $\mathbf{r_i}$ as defined in the previous section the following lemmas hold true.

Lemma 2:

$$\begin{aligned}(\mathbf{r_1}\times\mathbf{r_3})\cdot\mathbf{p_1} + (\mathbf{p_1}\times\mathbf{p_3})\cdot\mathbf{r_1} &= 0\\ (\mathbf{r_2}\times\mathbf{r_4})\cdot\mathbf{p_2} + (\mathbf{p_2}\times\mathbf{p_4})\cdot\mathbf{r_2} &= 0\end{aligned}$$

Proof: See [26]. ∎

Lemma 3:

$$\begin{aligned}(\mathbf{r_1}\times\mathbf{r_2})\cdot\mathbf{p_1} + (\mathbf{p_1}\times\mathbf{p_2})\cdot\mathbf{r_1} &= 0\\ (\mathbf{r_3}\times\mathbf{r_4})\cdot\mathbf{p_3} + (\mathbf{p_3}\times\mathbf{p_4})\cdot\mathbf{r_3} &= 0\end{aligned}$$

Proof: See [26]. ∎

The following lemmas show how vectors $\mathbf{m_i}$ are related to any finite rotation vector $\mathbf{v}\in\mathcal{L}$

Lemma 4: Let $\mathbf{v}\in\mathcal{L}$ be the finite rotation axis for a generic eye orientation. Then:

$$\begin{aligned}\mathbf{v} &\in span\{\mathbf{m_1},\mathbf{m_3}\}\\ \mathbf{v} &\in span\{\mathbf{m_2},\mathbf{m_4}\}\end{aligned}$$

Proof: (The first equality only is shown being the proof for the second identical.) Let us observe that $\mathbf{v} \in span\{\mathbf{m_1},\mathbf{m_3}\}$ is equivalent to condition $\mathbf{v}\cdot(\mathbf{m_1}\times\mathbf{m_3}) = 0$; then we have:

$$\begin{aligned}&\mathbf{v}\cdot(\mathbf{m_1}\times\mathbf{m_3}) =\\ =\ &\mathbf{v}\cdot[(\mathbf{r_1}\times\mathbf{p_1})\times(\mathbf{r_3}\times\mathbf{p_3})] =\\ =\ &\mathbf{v}\cdot\{[(\mathbf{r_1}\times\mathbf{p_1})\cdot\mathbf{p_3}]\ \mathbf{r_3} - [(\mathbf{r_1}\times\mathbf{p_1})\cdot\mathbf{r_3}]\ \mathbf{p_3}\}\end{aligned}$$

From the formula above and using equality (12) we obtain:

$$\begin{aligned} & \mathbf{v} \cdot (\mathbf{m_1} \times \mathbf{m_3}) = \\ = \quad & (\mathbf{v} \cdot \mathbf{p_3}) \, [(\mathbf{r_1} \times \mathbf{p_1}) \cdot \mathbf{p_3} - (\mathbf{r_1} \times \mathbf{p_1}) \cdot \mathbf{r_3}] = \\ = \quad & (\mathbf{v} \cdot \mathbf{p_3}) \, [(\mathbf{p_1} \times \mathbf{p_3}) \cdot \mathbf{r_1} + (\mathbf{r_1} \times \mathbf{r_3}) \cdot \mathbf{p_1}] = 0 \end{aligned}$$

where the last equality is due to lemma (2). ■

Lemma 5: Let $\mathbf{v} \in \mathcal{L}$ be the finite rotation axis for a generic eye orientation. Then:

$$\begin{aligned} \mathbf{v} &\in span\{\mathbf{m_1}, \mathbf{m_2}\} \\ \mathbf{v} &\in span\{\mathbf{m_1}, \mathbf{m_4}\} \\ \mathbf{v} &\in span\{\mathbf{m_2}, \mathbf{m_3}\} \\ \mathbf{v} &\in span\{\mathbf{m_3}, \mathbf{m_4}\} \end{aligned}$$

Proof: (The first relation only is shown being the proof for the others identical.) Let us observe that $\mathbf{v} \in span\{\mathbf{m_1}, \mathbf{m_2}\}$ is equivalent to condition $\mathbf{v} \cdot (\mathbf{m_1} \times \mathbf{m_2}) = 0$; then we have:

$$\begin{aligned} & \mathbf{v} \cdot (\mathbf{m_1} \times \mathbf{m_2}) = \\ = \quad & \mathbf{v} \cdot [(\mathbf{r_1} \times \mathbf{p_1}) \times (\mathbf{r_2} \times \mathbf{p_2})] = \\ = \quad & \mathbf{v} \cdot \{[(\mathbf{r_1} \times \mathbf{p_1}) \cdot \mathbf{p_2}] \, \mathbf{r_2} - [(\mathbf{r_1} \times \mathbf{p_1}) \cdot \mathbf{r_2}] \, \mathbf{p_2}\} \end{aligned}$$

Using equality (12) in the above formula we obtain:

$$\begin{aligned} & \mathbf{v} \cdot (\mathbf{m_1} \times \mathbf{m_2}) = \\ = \quad & (\mathbf{v} \cdot \mathbf{p_2}) \, [(\mathbf{r_1} \times \mathbf{p_1}) \cdot \mathbf{p_2} - (\mathbf{r_1} \times \mathbf{p_1}) \cdot \mathbf{r_2}] = \\ = \quad & (\mathbf{v} \cdot \mathbf{p_2}) \, [(\mathbf{p_1} \times \mathbf{p_2}) \cdot \mathbf{r_1} + (\mathbf{r_1} \times \mathbf{r_2}) \cdot \mathbf{p_1}] = 0 \end{aligned}$$

where the last equality is due to lemma (3). ■

It is now possible to show that for any eye orientation compatible with *Listing's Law* all the torque axes $\mathbf{m_i}$ belong to a common plane passing through the finite rotation axis $\mathbf{v} \in \mathcal{L}$.

Theorem 1: Let $\mathbf{v} \in \mathcal{L}$ be the finite rotation axis for a generic eye orientation, then there exists a plane $\mathcal{M}$ passing through $\mathbf{v}$ such that

$$\mathbf{m_i} \in \mathcal{M} \qquad \forall \, i = 1 \ldots 4$$

Proof: The proof follows from lemmas 4 and 5. ■

A second important result is that the relative position of the IPs and PPs, at any *Listing* compatible eye's orientation, form a set of parallel vectors, a stated by the following theorem.

Theorem 2: Let $\mathbf{v} \in \mathcal{L}$ be the finite rotation axis for a generic eye orientation, then:

$$(\mathbf{r_i} - \mathbf{p_i}) \times (\mathbf{r_j} - \mathbf{p_j}) = 0 \qquad \forall \, i, j = 1 \ldots 4$$

Proof: From theorem 1 vectors $\mathbf{v}$ and $\mathbf{m_i}$ belong to the same plane $\mathcal{M}$. Then vectors $\mathbf{v} \times \mathbf{m_i}$, $\forall \, i = 1 \ldots 4$ are all parallel and orthogonal to $\mathcal{M}$. In particular:

$$\begin{aligned} & \mathbf{v} \times \mathbf{m_i} = \\ = \quad & \mathbf{v} \times (\mathbf{r_i} \times \mathbf{p_i}) = \\ = \quad & \mathbf{v} \times [\mathbf{r_i} \times (\mathbf{p_i} - \mathbf{r_i})] = \\ = \quad & [\mathbf{v} \cdot (\mathbf{p_i} - \mathbf{r_i})]\mathbf{r_i} - (\mathbf{v} \cdot \mathbf{r_i})(\mathbf{p_i} - \mathbf{r_i}) \end{aligned}$$

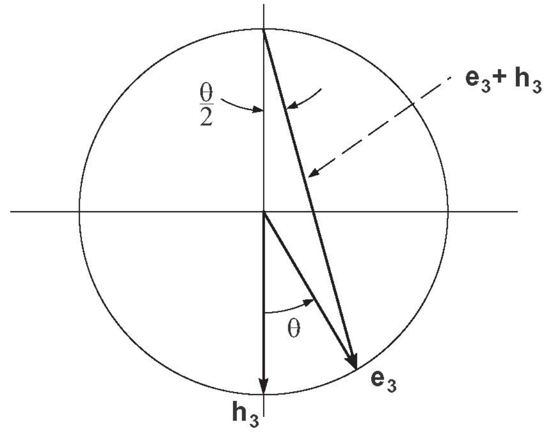

Fig. 4. Sketch of vectors $(\mathbf{r_0} - \mathbf{p_0})$, $\mathbf{r_0}$, and $\mathbf{h_3}$. The vector $\mathbf{v}$ is normal to the plane of the figure.

Using equality (12) the above equality can be rewritten as

$$\mathbf{v} \times \mathbf{m_i} = (\mathbf{v} \cdot \mathbf{r_i})(\mathbf{p_i} - \mathbf{r_i})$$

■

Consider now the following vector:

$$(\mathbf{r_0} - \mathbf{p_0}) = \sum_{i=1}^{4} (\mathbf{r_i} - \mathbf{p_i}), \tag{15}$$

from the above formula and by theorem 2, vector $(\mathbf{r_0} - \mathbf{p_0})$ is orthogonal to plane $\mathcal{M}$. Furthermore, by the assumptions on the symmetry of the vectors $\mathbf{p_i}$, and $\mathbf{c_i}$ we have:

$$\mathbf{p_0} = \sum_{i=1}^{4} \mathbf{p_i} = -\lambda \mathbf{h_3}, \tag{16}$$

and

$$\begin{aligned} \mathbf{r_0} \quad &= \quad \sum_{i=1}^{4} \mathbf{r_i} \; = \; \sum_{i=1}^{4} rot(\mathbf{v}, \theta) \; \mathbf{c_i} = \\ &= \quad rot(\mathbf{v}, \theta) \sum_{i=1}^{4} \mathbf{c_i} \; = \; \lambda \, rot(\mathbf{v}, \theta) \; \mathbf{h_3}, \end{aligned} \tag{17}$$

hence $\mathbf{r_0}$ is directed as $\mathbf{e_3}$.

Remark 1: The scalar λ depends on the placement of the IPs. □

It is finally possible to show that planes $\mathcal{M}$ and $\mathcal{P}_\omega$ are coincident.

Theorem 3: Let $\mathbf{v} \in \mathcal{L}$ be the finite rotation axis for a generic eye orientation, then:

$$\mathbf{m_i} \in \mathcal{M} \qquad \forall \, i = 1 \ldots 4$$

Proof: By expressions (15), (16), and (17), vector $(\mathbf{r_0} - \mathbf{p_0})$ forms an angle of $\frac{\theta}{2}$ with respect to axis $\mathbf{h_3}$, as sketched in Fig. 4. Then planes $\mathcal{M}$ and $\mathcal{P}_\omega$ pass through a common axis $\mathbf{v}$ and have the same normal. ■

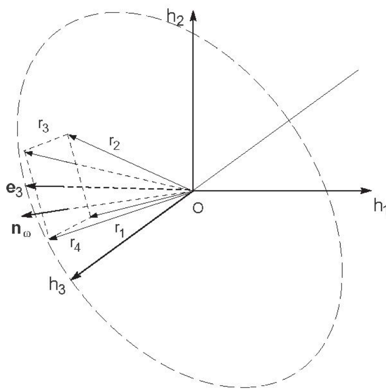

Fig. 5. When vector $\mathbf{n}_\omega$ belongs to the convex hull of vectors $\mathbf{r_i}$ then *rectii* EOMs can generate any admissible torque on $\mathcal{P}_\omega$.

Remark 2: Theorem (3) has in practice the following significant interpretation. For any Listing compatible eye orientation any possible moment applied to the eye, and generated using only the four *rectii* EOMs, must lay on plane $\mathcal{P}_\omega$. □

V. Reachability Issues

The main goal of this section is to show the Listing compatible eye orientations reachable from the *primary position*, using only four EOMs.

In the previous section it has been proved that for any Listing compatible orientation the vectors $\mathbf{m_i}$ span a unique plane. The problem now is to show, accordingly to formula (8), when arbitrary torques $\mathbf{m_i} \in \mathcal{P}_\omega$ can be generated using only pulling forces.

Theorem 2 states that $\mathbf{m_i}$ are all parallel to a vector $\mathbf{n}_\omega$ normal to $\mathcal{P}_\omega$. Therefore, formula (8) can be rewritten as:

$$\boldsymbol{\tau} = -\mathbf{n}_\omega \times \Big(\sum_{i=1}^{4} \gamma_i \; \mathbf{r_i} \Big) \tag{18}$$

where $\gamma_i = \frac{\tau_i}{|\mathbf{n}_\omega \times \mathbf{r_i}|} \geq 0$ take into account the actual EOMs pulling forces. In formula (18) is evident a *convex* linear combination of vectors $\mathbf{r_i}$. Then, it is possible to generate any torque vector laying on plane $\mathcal{P}_\omega$, as long as $\mathbf{n}_\omega$ belong to the convex hull of vectors $\mathbf{r_i}$, as shown in Fig. (5).

Remark 3: The discussion above shows that placement of the IPs affects the range of admissible motions of the eye. □

Accordingly with the previous discussion when the eye is in its *primary position* can be assigned any torque belonging to plane $\mathcal{L}$. Assume now that, under the assumptions made in section III, a simplified dynamic model of the eye could be expressed as:

$$I\dot{\boldsymbol{\omega}} = \boldsymbol{\tau} \tag{19}$$

where I is the inertia matrix the eye, assumed to be diagonal. Assume the eye to be in the *primary position*, with zero angular velocity (zero state). The EOMs can generate a resulting moment of the form:

$$\boldsymbol{\tau} = \mathbf{v}\theta(t) \tag{20}$$

where $\mathbf{v} \in \mathcal{L}$ is a constant vector and $\theta(t)$ a scalar control signal. Therefore, $\dot{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}$ are parallel to $\mathbf{v}$. Then, it is possible to reach any Listing compatible orientation, and also, during the rotation, the *Half Angle Rule* is satisfied.

Similar reasoning can be applied to control the eye orientation to the *primary position* starting from any Listing compatible orientation and zero angular velocity.

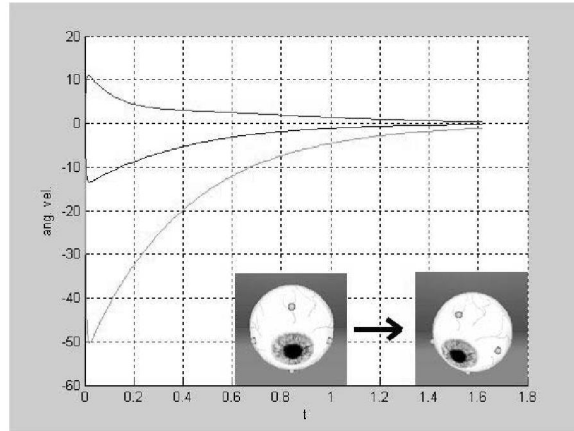

Fig. 6. Components of angular velocity during a saccade from a secondary to a tertiary position.

VI. Simulations and Experimental Tests

Further analysis beyond the analytical results presented in this paper has been carried on using simulation tools. Tests have been done assuming viscoelastic actuation forces generated by the EOMs. In particular, Fig. 6 shows the components of the angular velocity for a generic saccade from a secondary to a tertiary position. Fig. 7 shows the component of the vector $\mathbf{v}$ off the Listing's plane, which is clearly negligible also for a generic saccade.

The analytical and simulative results previously discussed provided the support for the development of a tendon driven robot eye (MAC-EYE), [25]. Fig. (8) shows the complete system including the embedded control electronics.

Each *eye* is actuated by four independent DC motors driving tendons routed to the eye-ball. Internal custom optical sensors provide feedback to control the mechanical tension of the tendons. The eye- ball is made of PTFE and is supported by a custom PTFE bearing. Sliding pulleys emulating the geometry of the PPs discussed in the paper have been implemented as discussed in [25], [26].

VII. Conclusions

In this paper we have investigated the possibility of emulating the actual saccadic motions implementing the *Listing's Law* on a mechanical basis. To this aim, a model of the eye plant has been proposed. The model is characterized by

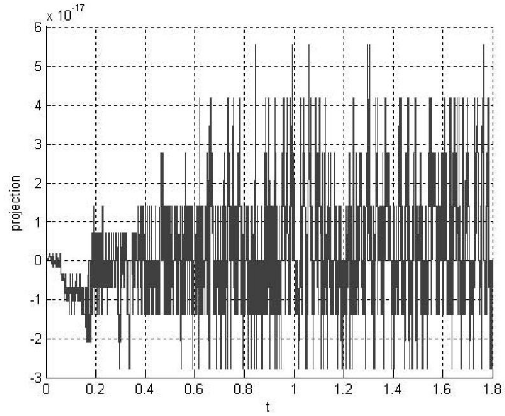

Fig. 7. Component of the rotation vector off the *Listing's Plane*.

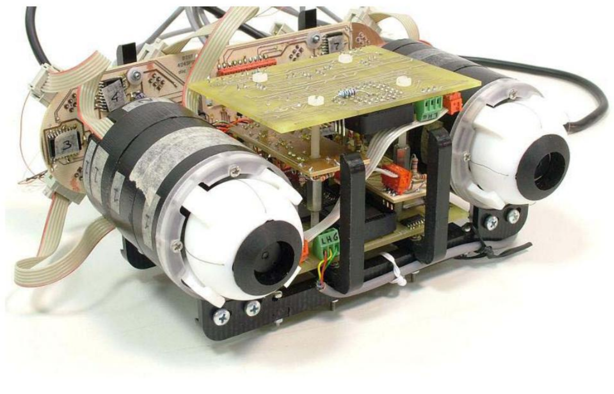

Fig. 8. Complete stereoscopic robot system

the relative position of the IPs on the eye-ball and of the PPs required to properly route the EOMs. Simple geometric conditions on these quantities allow to constrain the space of the moments generated by the action of the EOMs to a single plane coincident with plane $\mathcal{P}_\omega$. This property allows to show that any (reachable) Listing compatible eye orientation can be reached from the *primary position* with a trajectory composed by Listing compatible eye orientations.

Numerical simulations suggest that similar results should be valid also for generic saccades.

Experimental tests are currently ongoing to validate the model on a prototype tendo driven robot.

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